Exercise Sheet Solutions #4

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P1. Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f : X \to \mathbb{C}$ be a measurable function. Then prove that $f = 0$ a.e. if and only if $\int_E f d\mu = 0$ for every measurable set $E \in \mathcal{B}$.

> ***Solution:*** If $f = 0$ a.e. it follows that $f\mathbb{1}_E = 0$ a.e. and thus
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> $$\int_E f\mu = \int_X \mathbb{1}_E f d\mu = 0.$$
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> Let us assume now that $\int_E f d\mu = 0$ for every measurable set $E \in \mathcal{B}$. As
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> $$0 = \int_E f d\mu = \int_E Re(f) d\mu + i \int_E Im(f) d\mu \Rightarrow 0 = \int_E Re(f) d\mu = \int_E Im(f) d\mu,$$
>
> whence it follows that we can assume without loss of generality that $f : X \to \mathbb{R}$. Take $E = \{x \in X \mid f(x) > 0\}$. We have that $f\mathbb{1}_E \geq 0$ a.e.. Besides
>
> $$0 = \int_E f d\mu = \int \mathbb{1}_E f d\mu,$$
>
> which implies that $\mathbb{1}_E f = 0$ a.e.. By a similar argument with the set E^c we get that $\mathbb{1}_{E^c} f = 0$ a.e., which concludes that $f = 0$ a.e..

P2. Prove that a space is complete if and only if the following property holds: for functions $f, g; X \to \mathbb{C}$, if f is measurable and $f = g$ a.e., then g is also measurable.

> ***Solution:*** Assume that the space is complete. Let $B \subseteq \mathbb{C}$ an open set. Then, we notice that $g^{-1}(B) \setminus f^{-1}(B), f^{-1}(B) \setminus g^{-1}(B) \subseteq \{x \in X \mid f(x) \neq g(x)\}$ which has measure 0. As the space is complete, we have that $g^{-1}(B) \setminus f^{-1}(B)$ and $f^{-1}(B) \setminus g^{-1}(B)$ are measurable. The latter set being measurable implies that $(g^{-1}(B) \cap f^{-1}(B))$ is measurable due to
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> $$(g^{-1}(B) \cap f^{-1}(B)) = f^{-1}(B) \setminus (f^{-1}(B) \setminus g^{-1}(B)).$$
>
> Thus we have that the set $g^{-1}(B) = (g^{-1}(B) \setminus f^{-1}(B)) \cup (g^{-1}(B) \cap f^{-1}(B))$ is measurable, and thus g is measurable by the fact that B was arbitrary. For the other direction, let $A \subseteq X$ be a measurable set of measure 0. Let $B \subseteq A$. Then $\mathbb{1}_A = \mathbb{1}_B$ a.e., which implies that $\mathbb{1}_B$ is measurable, and thus B is in the sigma algebra. In this way, we conclude the completeness.

P3. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{B} : \mu(N) = 0\}$ be the σ-ideal of μ-null sets. Show that the family $\overline{\mathcal{B}} = \{E \cup F : E \in \mathcal{B}, F \subseteq N \in \mathcal{N}\}$ is a σ-algebra, and there is a unique extension $\bar{\mu}$ of μ to $\overline{\mathcal{B}}$.

> ***Solution:*** First, we see that $\overline{\mathcal{B}}$ is a sigma algebra. Indeed, we have that $X \in \mathcal{B} \subseteq \overline{\mathcal{B}}$. On the other hand, if $E \cup F \in \overline{\mathcal{B}}$ with $E \in \mathcal{B}$ and $F \subseteq N \in \mathcal{N}$, we have that
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> $$(E \cup F)^c = E^c \cap F^c = E^c \cap N^c \cup (E^c \cap F^c \setminus N^c),$$
>
> in where the first set is measurable and the second is contained in the set N which has measure 0. Hence, $(E \cup F)^c \in \mathcal{B}$.
>
> Now, we take $E_i \cup F_i \in \overline{\mathcal{B}}$ with $E_i \in \mathcal{B}$ and $F_I \subseteq N_i \in \mathcal{N}$, for $i \in \{1, 2\}$. Then, givem that

$E_1 \cup E_2 \in \mathcal{B}$, $F_1 \cup F_2 \subseteq N_1 \cup N_2 \in \mathcal{N}$, we have that

$$(E_1 \cup F_1) \cup (E_2 \cup F_2) = (E_1 \cup E_2) \cup (F_1 \cup F_2) \in \overline{\mathcal{B}},$$

concluding thus that $\overline{\mathcal{B}}$ is a sigma algebra. Finally, we define $\overline{\mu}(E \cup F) := \mu(E)$. This clearly defines a measure (check this! it should follow from elementary properties). This extension is unique by the fact that any extension of μ to $\overline{\mathcal{B}}$ must give sets $F \subseteq N \in \mathcal{N}$ measure 0, thus if ν is one of those extensions: $\nu(E \cup F) = \nu(E) = \mu(E)$.

P4. Let $(X, \mathcal{B}, \mu)$ be a measure space. Prove that simple functions are dense in $L^1(X)$.

Solution: Let $f \in L^1(X)$ taking positive values. The, we already know that there are simple functions $(f_n)_n \subseteq L^1(X)$ such that $0 \leq f_n \leq f$ and $f_n \nearrow f$. Thus

$$\|f - f_n\|_1 = \int |f - f_n| d\mu = \int f - f_n d\mu \to 0.$$

Now, for a function f taking real values in $L^1(X)$, we divide $f = f_+ - f_-$. Let $(f_n)_n$ and $(g_n)_n$ sequences of simple functions such that $f_n \to f_+$ and $g_n \to f_-$. Then the sequence of simple functions $(f_n - g_n)_n$ is such that

$$\|f - (f_n - g_n)\|_1 = \|(f_+ - f_n) - (f_- - g_n)\|_1 \leq \|(f_+ - f_n)\|_1 + \|(f_- - g_n)\|_1 \to 0 \text{ as } n \to \infty.$$

A similar argument gives the approximation of a complex function f, by dividing $f = Re(f) + iIm(f)$, concluding that simple functions are dense in $L^1(X)$.

P5. Let $(X, \mathcal{B}, \mu)$ be a measure space and $E \in \mathcal{B}$ a set of positive measure. If $\{E_n\}_{n \in \mathbb{N}}$ is an increasing sequence of measurable sets and $E = \bigcup_{n \in \mathbb{N}} E_n$ prove that for every function $f \in L^1(X)$

$$\lim_{n \to \infty} \int_{E_n} f d\mu = \int_E f d\mu$$

and state and prove an analogous result for decreasing sequences.

Solution: We notice that $\mathbb{1}_{E_n} \to \mathbb{1}_E$ a.e., and thus $\mathbb{1}_{E_n} f \to \mathbb{1}_E f$. This in addition to $|\mathbb{1}_{E_n} f| \leq |f|$ and the fact that $|f|$ is integrable, implies by dominated convergence theorem, we have that

$$\lim_{n \to \infty} \int_{E_n} f d\mu = \lim_{n \to \infty} \int \mathbb{1}_{E_n} f d\mu = \int \mathbb{1}_E f = \int_E f d\mu,$$

concluding.

P6. Prove (yet again) the identity

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \sum_{k \in \mathbb{N}_0} \frac{x^k}{k!}.$$

Solution: We first notice that

$$\left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{n^k} \tag{1}$$

Define the function

$$f_n(k) = \begin{cases} \binom{n}{k}\frac{x^k}{n^k} & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases}.$$

Consider the measure ν on $\mathbb{N}_0$ such that $\nu(A) = |A|$ for each $A \subseteq \mathbb{N}_0$ (we dote $\mathbb{N}_0$ of the discrete sigma algebra, i.e. $\mathcal{P}(\mathbb{N})$). Then, we have that

$$\int f_n d\nu = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{n^k}.$$

On other hand, notice that

$$f_n(k) = \binom{n}{k}\frac{x^k}{n^k} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{n^k}\frac{x^k}{k!} \to \frac{x^k}{k!} \text{ as } n \to \infty$$

for each k, as well as

$$|f_n(k)| \leq |\frac{x^k}{k!}|,$$

where the left-hand side is integrable for ν. Indeed, if $f(k) = \frac{x^k}{k!}$ we have that

$$\int |f| d\nu = \sum_{k=0}^{\infty} \frac{|x|^k}{k!} = e^{|x|} < \infty.$$

Consequently, by DCT, we have that $f_n \to f$ in $L^1(\nu)$, which implies that

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty}\int f_n d\nu = \int f d\nu = \sum_{k\in\mathbb{N}_0}\frac{x^k}{k!}.$$